

Test 5

Section I, Part A

Allotted Time: 62 Minutes

Number of Questions: 29 Questions

Calculator Utilization: The use of a calculator is not permitted for this part of the test.

Instructions: Work through each of the problems, using the space provided to jot down calculations or notes. Carefully examine the answer choices for each question and determine which is the most accurate. Once you've selected the best answer, fill in the matching oval on the answer sheet. Keep in mind that anything written in this booklet won't be graded. Don't get stuck on any single question for too long. In this test, unless stated otherwise, assume that the domain of a function f includes all real values of x that result in $f(x)$ being a real number:

1. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} =$

- a. 0
- b. 3
- c. 6
- d. The limit does not exist

2. If $f(x) = 5x^4 - 3 \sin x + 2 \cos x$, what is $f'(x)$?

- a. $20x^3 - 3 \cos x - 2 \sin x$
- b. $20x^3 + 3 \cos x - 2 \sin x$
- c. $5x^3 - 3 \cos x + 2 \sin x$
- d. $20x^3 - 3 \sin x - 2 \cos x$

3. The derivative of a function f is

$$f'(x) = (x + 2)(x - 1)^2.$$

Which of the following statements is true?

- a. f has a local maximum at $x = -2$ and a local minimum at $x = 1$.
- b. f has a local minimum at $x = -2$ and no local extremum at $x = 1$.
- c. f has no local extremum at $x = -2$ and a local maximum at $x = 1$.
- d. f has local minima at both $x = -2$ and $x = 1$.

4. Let

$$y = (3x^2 - 1)^5.$$

Find $\frac{dy}{dx}$.

- a. $5(3x^2 - 1)^4$
- b. $10x(3x^2 - 1)^4$
- c. $30x(3x^2 - 1)^4$
- d. $15x(3x^2 - 1)^5$

5. Find

$$\int (4x^3 + e^x - 2 \cos x) \, dx.$$

- a. $x^4 + e^x - 2 \sin x + C$
- b. $12x^2 + e^x + 2 \sin x + C$
- c. $x^4 + e^x + 2 \sin x + C$
- d. $4x^4 + e^x - 2 \sin x + C$

6. Let

$$f(x) = \begin{cases} kx + 1, & x < 2 \\ x^2 - 3, & x \geq 2 \end{cases}$$

What value of k makes f continuous at $x = 2$?

- a. -1
- b. 0
- c. 1
- d. 2

7. The position of a particle moving along a line is given by

$$s(t) = t^3 - 6t^2 + 9t$$

in meters, where t is measured in seconds. What is the instantaneous velocity of the particle at $t = 2$?

- a. -3 m/s
- b. 0 m/s
- c. 3 m/s
- d. 9 m/s

8. Suppose

$$f''(x) = (x - 1)(x + 3).$$

Which statement about the concavity of f is true?

- a. f is concave up on $(-3, 1)$ and concave down on $(-\infty, -3) \cup (1, \infty)$.
- b. f is concave up on $(-\infty, -3) \cup (1, \infty)$ and concave down on $(-3, 1)$.
- c. f is concave up on $(-\infty, 1)$ and concave down on $(1, \infty)$.
- d. f is concave up on $(-\infty, -3)$ and concave down on $(-3, \infty)$.

9. If

$$g(x) = x^2 e^x,$$

Then $g'(x) =$

- a. $2xe^x$
- b. $x^2 e^x$
- c. $e^x(x^2 + 2x)$
- d. $e^x(x^2 - 2x)$

10. Find the area of the region bounded by the graph of
 $y = x^2 - 1$
and the x -axis on the interval $[-2, 2]$.

- a. $\frac{4}{3}$
- b. $\frac{8}{3}$
- c. $\frac{10}{3}$
- d. 4

11. The graph of f consists of line segments from $(0, 0)$ to $(2, 4)$, from $(2, 4)$ to $(5, 4)$, and from $(5, 4)$ to $(7, 0)$. What is

$$\int_0^7 f(x) \, dx?$$

- a. 16
- b. 20
- c. 24
- d. 28

12. The curve is defined by

$$x^2 + 2xy + y^2 = 16.$$

What is $\frac{dy}{dx}$ at the point $(2, 2)$?

- a. -1
- b. $-\frac{1}{2}$
- c. $\frac{1}{2}$
- d. 1

13. Let $f(x) = x^2 - 2x + 4$ on $[1, 5]$. Find the value of c that satisfies the conclusion of the Mean Value Theorem.

- a. 1
- b. 2
- c. 3
- d. 4

14. Let

$$f(x) = \begin{cases} 2x + 1, & x < 1 \\ 4, & x = 1 \\ x^2 + 2, & x > 1 \end{cases}$$

Which statement is true?

- a. $\lim_{x \rightarrow 1} f(x)$ does not exist.
- b. $\lim_{x \rightarrow 1} f(x) = 3$, and f is continuous at $x = 1$.
- c. $\lim_{x \rightarrow 1} f(x) = 4$, and f is continuous at $x = 1$.
- d. $\lim_{x \rightarrow 1} f(x) = 3$, but f is not continuous at $x = 1$.

15. Let

$$G(x) = \int_0^{x^2} \cos t \, dt.$$

What is $G'(x)$?

- a. $\cos(x^2)$
- b. $2x \cos(x)$
- c. $2x \cos(x^2)$
- d. $\sin(x^2)$

16. A circle's radius is increasing at a rate of 3 cm/s. At the instant when the radius is 4 cm, how fast is the area of the circle increasing?

- a. $12\pi \text{ cm}^2/\text{s}$
- b. $16\pi \text{ cm}^2/\text{s}$
- c. $24\pi \text{ cm}^2/\text{s}$
- d. $48\pi \text{ cm}^2/\text{s}$

17. For the differential equation $\frac{dy}{dx} = x + y$, at which point is the slope of the solution curve equal to 0?

- a. $(1, 1)$
- b. $(1, -1)$
- c. $(0, 1)$
- d. $(-1, -1)$

18. If $h(x) = x^2 \ln(5x^3)$, then $h'(x) =$

- a. $2x \ln(5x^3) + 3x$
- b. $2x \ln(5x^3) + x^2$
- c. $x^2 \cdot \frac{3}{x}$
- d. $2x \ln(5x^3) + 15x^4$

19. Find the average value of

$$f(x) = x^2 + 1$$

on the interval $[0, 3]$.

- a. 3
- b. 4
- c. 5
- d. 6

20. Let $f(x) = x^3 - 6x^2 + 9x + 1$ on $[0, 4]$. What is the absolute maximum value of f on the interval?

- a. 1
- b. 3
- c. 4
- d. 5

21. Evaluate

$$\int_0^1 6x(1 + 3x^2)^4 dx.$$

- a. $\frac{255}{4}$
- b. $\frac{256}{5}$
- c. $\frac{1024}{5}$
- d. $\frac{1023}{5}$

22. If $f(2) = 5$ and $f'(2) = -3$, what is $(f^{-1})'(5)$?

- a. -3
- b. $-\frac{1}{3}$
- c. $\frac{1}{3}$
- d. 3

23. Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin(4x)}{3x}.$$

- a. $\frac{4}{3}$
- b. $\frac{3}{4}$
- c. 1
- d. 0

24. The graph of $f'(x)$ has the following sign behavior:

- $f'(x) < 0$ for $x < -1$
- $f'(-1) = 0$
- $f'(x) > 0$ for $-1 < x < 2$
- $f'(2) = 0$
- $f'(x) > 0$ for $2 < x < 4$
- $f'(4) = 0$
- $f'(x) < 0$ for $x > 4$

Which statement is true?

- a. f has local maxima at $x = -1$ and $x = 4$, and no extremum at $x = 2$.
- b. f has local minima at $x = -1$ and $x = 2$, and a local maximum at $x = 4$.
- c. f has a local minimum at $x = -1$, no local extremum at $x = 2$, and a local maximum at $x = 4$.
- d. f has no local extrema.

25. Which definite integral is represented by

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{4i}{n}\right)^3 \cdot \frac{4}{n} ?$$

- a. $\int_0^4 (1 + 4x)^3 dx$
- b. $\int_1^4 x^3 dx$
- c. $\int_0^1 (1 + 4x)^3 dx$
- d. $\int_0^4 (1 + x)^3 dx$

26. A particle moves along a line with velocity

$$v(t) = t^2 - 5t + 4$$

and acceleration

$$a(t) = 2t - 5$$

for $0 < t < 6$. On which interval(s) is the particle speeding up?

- a. $(0, 1)$ only
- b. $\left(1, \frac{5}{2}\right)$ and $(4, 6)$
- c. $\left(\frac{5}{2}, 4\right)$ only
- d. $(0, 1)$ and $\left(\frac{5}{2}, 4\right)$

27. Which function satisfies

$$\frac{dy}{dx} = 2x(y + 1)$$

And the initial condition $y(0) = 0$?

- a. $y = e^{x^2} - 1$
- b. $y = e^{2x} - 1$
- c. $y = 2e^{x^2} - 1$
- d. $y = x^2 + 1$

28. The region bounded by $y = x^2$, $y = 0$, and $x = 3$ is revolved about the x -axis. What is the volume of the resulting solid?

- a. $\frac{81\pi}{5}$
- b. $\frac{243\pi}{5}$
- c. 27π
- d. 81π

29. The function f is differentiable, and values of $f(x)$ near $x = 2$ are shown below.

x	1.9	2.0	2.1
$f(x)$	6.61	7.00	7.41

Which of the following is the best estimate of $f'(2)$?

- a. 3.8
- b. 4.0
- c. 4.2
- d. 7.0

Section I, Part B

Allotted Time: 38 Minutes

Number of Questions: 13 Questions

Calculator Utilization: The use of a calculator is permitted for the questions in this part of the test.

Instructions: For each problem below, solve using the space provided for scratch work. Review the format of the answer options, select the best choice, and mark the corresponding oval on your answer sheet. Note that any work written in the test booklet will not receive credit, so focus on completing the answer sheet. Avoid spending excessive time on any single question. The exact numerical answer may not always be included in the answer choices. In such cases, select the option that most closely approximates the exact value. Unless stated otherwise, the domain of a function is assumed to include all real numbers x for which $f(x)$ produces a real number.

30. Values of a continuous function f are shown below.

x	0	1	2.5	4	6	7
$f(x)$	4.2	5.0	6.3	5.8	4.6	3.9

Using the trapezoidal rule with the intervals shown in the table, approximate $\int_0^7 f(x) \, dx$.

- a. 34.6
- b. 36.8
- c. 38.2
- d. 40.1

31. The curve is defined implicitly by

$$x^2 + xy + 3y^2 = 12.$$

At the point $(1.4, 1.7)$, approximate $\frac{dy}{dx}$.

- a. -0.39
- b. -0.74
- c. 0.39
- d. 0.74

32. Let $g(x) = \sin 2x + \ln(x + 3) - 0.4x$ on $[0, 4]$. Use a calculator to determine the absolute maximum value of g on $[0, 4]$.

- a. 1.10
- b. 1.34
- c. 2.02
- d. 2.71

33. A particle's position $s(t)$, in meters, is shown in the table below:

t	4.8	4.9	5.0	5.1	5.2
$s(t)$	20.12	21.03	21.98	22.97	24.00

What is the best estimate of $s'(5)$.

- a. 8.7
- b. 9.7
- c. 10.3
- d. 11.0

34. Values of a function $h(x)$ near $x = 1$ are shown.

x	0.99	0.999	1.001	1.01
$h(x)$	2.81	2.98	4.02	4.19

Based on the table, which statement is most accurate?

- a. $\lim_{x \rightarrow 1} h(x) = 3$
- b. $\lim_{x \rightarrow 1} h(x) = 4$
- c. $\lim_{x \rightarrow 1} h(x)$ does not exist
- d. $h(1) = 3.5$

35. Let

$$A(x) = \int_0^x (e^{-t^2/2} + \sin t) dt.$$

Use a calculator to approximate $A(3)$.

- a. 2.51
- b. 3.24
- c. 3.89
- d. 4.36

36. The region enclosed by $y = \ln(x + 1)$ and $y = \frac{x}{3}$ for $x \geq 0$ is bounded by their intersection at $x = 0$ and one additional positive intersection. Use a calculator to find the area of the region.

- a. 1.18
- b. 1.63
- c. 2.11
- d. 2.74

37. A function y satisfies

$$\frac{dy}{dx} = \frac{x^2}{1 + y}, \quad y(0) = 1.$$

Find $y(2)$. Round to the nearest hundredth.

- a. 1.62
- b. 1.84
- c. 2.06
- d. 2.31

38. Let

$$f(x) = e^{-x} \sin(x^2).$$

Use a calculator to approximate $f'(1.5)$.

- a. -0.59
- b. -0.21
- c. 0.21
- d. 0.59

39. A twice-differentiable function f has

$$f''(x) = e^{-x} - 0.2x.$$

Use a calculator to approximate the x -value where f has a point of inflection on $[0, 5]$.

- a. 0.84
- b. 1.33
- c. 2.10
- d. 3.47

40. The effectiveness of a medication, in arbitrary units, t hours after it is taken is modeled by

$$E(t) = \frac{50t}{t+2} - 0.5t$$

for $0 \leq t \leq 20$. Use a calculator to determine when the medication is most effective.

- a. 8.63 hours
- b. 10.00 hours
- c. 12.14 hours
- d. 15.28 hours

41. Find the average value of

$$f(x) = \sqrt{1+x^3}$$

on the interval $[0, 2]$.

- a. 1.21
- b. 1.62
- c. 2.04
- d. 2.71

42. The volume V of a balloon depends on its radius r , and the radius depends on time t . Suppose

$$\frac{dV}{dr} = 48\pi$$

when $r = 4$, and

$$\frac{dr}{dt} = 0.25$$

when $t = 6$. What is $\frac{dV}{dt}$ at $t = 6$?

- a. 8π
- b. 12π
- c. 16π
- d. 48π

Section II, Part A

Allotted Time: 30 Minutes

Number of Questions: 2 Questions

Calculator Utilization: The use of a calculator is permitted for the questions in this part of the test.

Instructions: Providing your set up for these problems is essential, as partial credit may be granted. If you include decimal approximations, ensure they are accurate to three decimal places.

1. A company models the daily revenue, in thousands of dollars, from selling x hundred items by

$$R(x) = 12x + 8 \ln(x + 1) - 0.4x^2$$

for $0 \leq x \leq 20$.

- Find $R'(x)$.
 - Use a calculator to find the value of x in $[0, 20]$ where $R(x)$ is maximized. Give your answers to three decimal places.
 - Find the maximum revenue, in dollars, predicted by the model. Round to the nearest dollar.
 - Revenue is increasing at a decreasing rate when $R'(x) > 0$ and $R''(x) < 0$. Determine the interval(s) in $[0, 20]$ where revenue is increasing at a decreasing rate. Justify your answer.
2. A reservoir contains 5,000 gallons of water at time $t = 0$. Water is leaking out of the reservoir at a rate of $L(t)$ gallons per hour. Selected values of $L(t)$ are shown in the table below.

t (hours)	0	1	2.5	4	6	7	9
$W(t)$ (gal/hr)	120	135	160	150	110	95	80

- a. Use the trapezoidal rule with the data in the table to approximate

$$\int_0^9 L(t) dt.$$

- b. Interpret the meaning of

$$\int_0^9 L(t) dt$$

in the context of the problem. Include units.

- Find the average rate at which water leaks out of the reservoir over the interval $0 \leq t \leq 9$. Include units.
- Use your answer from part (a) to approximate the amount of water in the pond at time $t = 9$.

Section II, Part B

Allotted Time: 60 Minutes

Number of Questions: 4 Questions

Calculator Utilization: The use of a calculator is not permitted for this part of the test.

3. Let f be the piecewise function

$$f(x) = \begin{cases} ax^2 + bx + 1, & x \leq 1 \\ 3\sqrt{x+3} + c, & x > 1 \end{cases}$$

where a , b , and c are all constants.

- Find the value of c in terms of a and b so that f is continuous at $x = 1$.
- Find a relationship between a and b so that f is differentiable at $x = 1$.
- Assuming f is continuous and differentiable at $x = 1$, write the equation of the tangent line to f at $x = 1$.
- Assume that f is differentiable at $x = 1$. If $a = 2$, find the values of b and c .

4. Let

$$f(x) = x^3 - 3x^2 - 9x + 4.$$

- Find $f'(x)$ and determine all critical values of f .
- On which intervals is f increasing? On which intervals is f decreasing? Justify your answer.
- Find all x -values where f has a point of inflection. Justify your answer.
- Verify that the hypotheses of the Mean Value Theorem are satisfied for f on the interval $[0, 4]$. Then find all values of c in $(0, 4)$ that satisfy the conclusion of the Mean Value Theorem.

5. Let f be a continuous function defined on $[0, 7]$. The graph of f is made of the following pieces:
- From $x = 0$ to $x = 2$, f is a line segment from $(0, 3)$ to $(2, -1)$.
 - From $x = 2$ to $x = 5$, f is a line segment from $(2, -1)$ to $(5, -1)$.
 - From $x = 5$ to $x = 7$, f is a line segment from $(5, -1)$ to $(7, 3)$.

Define

$$g(x) = \int_0^x f(t) dt.$$

- Find $g(2)$.
 - Find $g'(6)$.
 - On which interval(s) in $[0, 7]$ is g decreasing? Justify your answer.
 - Find the absolute minimum value of $g(x)$ on $[0, 7]$. Justify your answer.
6. A function $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = x(y - 2)$$

with initial condition

$$f(0) = 5.$$

- Show that $y = 2 + 3e^{x^2/2}$ is a solution to the differential equation with the given initial condition.
- Find the general solution to the differential equation.
- Find the particular solution that satisfies $f(0) = 5$ and determine $f(2)$.
- For the particular solution from part (c), determine the interval(s) of x for which the solution is increasing. Justify your answer.

Solutions

Section I, Part A

1. C	2. A	3. B	4. C	5. A
6. B	7. A	8. B	9. C	10. D
11. B	12. A	13. C	14. D	15. C
16. C	17. B	18. A	19. B	20. D
21. D	22. B	23. A	24. C	25. D
26. B	27. A	28. B	29. B	

1. Factor the numerator:

$$x^2 - 9 = (x - 3)(x + 3).$$

For $x \neq 3$,

$$\frac{x^2 - 9}{x - 3} = x + 3.$$

So, the limit is

$$\lim_{x \rightarrow 3} x + 3 = 3 + 3 = 6$$

2. Differentiate term-by-term:

$$\frac{d}{dx}(5x^4) = 20x^3, \quad \frac{d}{dx}(-3 \sin x) = -3 \cos x, \quad \frac{d}{dx}(2 \cos x) = -2 \sin x$$

3. Since $(x - 1)^2 \geq 0$, the sign of $f'(x)$ is determined by $(x + 2)$, except at $x = 1$ where the derivative is zero but does not change sign. Thus f' changes from negative to positive at $x = -2$, giving a local minimum, and there is no extremum at $x = 1$.

4. Use the chain rule:

$$\frac{dy}{dx} = 5(3x^2 - 1)^4(6x) = 30x(3x^2 - 1)^4.$$

5. Integrate term-by-term:

$$\int 4x^3 dx = x^4, \quad \int e^x dx = e^x, \quad \int -2 \cos x dx = -2 \sin x.$$

6. Continuity at $x = 2$ requires

$$2k + 1 = 2^2 - 3 = 1.$$

So,

$$2k = 0 \Rightarrow k = 0.$$

7. Velocity is the derivative of position:

$$v(t) = s'(t) = 3t^2 - 12t + 9.$$

Then:

$$v(2) = 3(4) - 12(2) + 9 = 12 - 24 + 9 = -3.$$

8. Set $f''(x) = 0$, giving $x = -3$ and $x = 1$. Testing intervals shows $f'' > 0$ on $(-\infty, -3)$ and $(1, \infty)$, and $f'' < 0$ on $(-3, 1)$.

9. Using the product rule:

$$g'(x) = 2xe^x + x^2e^x = e^x(x^2 + 2x).$$

10. The function crosses the x -axis at $x = -1$ and $x = 1$. The area is

$$\int_{-2}^{-1} (x^2 - 1) dx + \int_{-1}^1 (1 - x^2) dx + \int_1^2 (x^2 - 1) dx.$$

By symmetry,

$$2 \int_1^2 (x^2 - 1) dx + \int_{-1}^1 (1 - x^2) dx.$$

$$2 \left[\frac{x^3}{3} - x \right]_1^2 + \left[x - \frac{x^3}{3} \right]_{-1}^1 = 2 \left(\frac{4}{3} \right) + \frac{4}{3} = \frac{12}{3} = 4.$$

11. The area is a left triangle, a rectangle, and a right triangle:

$$\frac{1}{2}(2)(4) + 3(4) + \frac{1}{2}(2)(4) = 4 + 12 + 4 = 20.$$

12. Differentiate implicitly:

$$2x + 2 \left(x \frac{dy}{dx} + y \right) + 2y \frac{dy}{dx} = 0.$$

So,

$$2x + 2y + (2x + 2y) \frac{dy}{dx} = 0.$$

At $(2, 2)$:

$$4 + 4 + (4 + 4) \frac{dy}{dx} = 0$$

$$8 + 8 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -1.$$

13. Average rate of change:

$$\frac{f(5) - f(1)}{5 - 1} = \frac{(25 - 10 + 4) - (1 - 2 + 4)}{4} = \frac{19 - 3}{4} = 4.$$

Since $f'(x) = 2x - 2$, set $2c - 2 = 4$, then solve for c .

$$2c - 2 = 4 \Rightarrow 2c = 6 \Rightarrow c = 3.$$

14. Left-hand limit $= 2(1) + 1 = 3$. Right-hand limit $= (1)^2 + 2 = 3$. So, the two-sided limit is 3.

But $f(1) = 4$, so f is not continuous at $x = 1$.

15. By the Fundamental Theorem of Calculus and the chain rule,

$$G'(x) = \cos(x^2) \cdot 2x.$$

16. Since

$$A = \pi r^2,$$

differentiate with respect to t :

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

At $r = 4$ and $\frac{dr}{dt} = 3$,

$$\frac{dA}{dt} = 2\pi(4)(3) = 24\pi.$$

17. Slope 0 occurs where $\frac{dy}{dx} = 0 \Rightarrow x + y = 0 \Rightarrow y = -x$. Only $(1, -1)$ fits this requirement.

18. By applying product rule and chain rule:

$$h'(x) = \frac{d}{dx}(x^2) \cdot \ln(5x^3) + x^2 \cdot \frac{d}{dx}[\ln(5x^3)]$$

$$h'(x) = 2x \cdot \ln(5x^3) + x^2 \cdot \frac{1}{5x^3}(15x^2)$$

$$h'(x) = 2x \cdot \ln(5x^3) + x^2 \cdot \frac{3}{x}$$

$$h'(x) = 2x \ln(5x^3) + 3x$$

19. Average value:

$$\frac{1}{3-0} \int_0^3 (x^2 + 1) dx = \frac{1}{3} \left[\frac{x^3}{3} + x \right] \bigg|_0^3 = \frac{1}{3}(9 + 3) = 4.$$

20. First, find the critical values of f on the given interval $[0, 4]$:

$$f'(x) = 3x^2 - 12x + 9 = 0 \Rightarrow 3(x - 1)(x - 3) = 0.$$

So, $x = 1$ and $x = 3$. Then, evaluate f at the critical values and endpoints of the interval.

$$f(0) = 1, \quad f(1) = 5, \quad f(3) = 1, \quad f(4) = 5.$$

So, the maximum value on the interval is 5.

21. Let $u = 1 + 3x^2$, so $du = 6x dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 4$. Thus

$$\int_0^1 6x(1 + 3x^2)^4 dx = \int_1^4 u^4 du = \left[\frac{u^5}{5} \right]_1^4 = \frac{4^5 - 1}{5} = \frac{1023}{5}.$$

22. Applying the derivative of an inverse formula:

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(2)} = \frac{1}{-3} = -\frac{1}{3}$$

23. This problem can be solved using L'Hospital's Rule. Alternatively:

$$\frac{\sin(4x)}{3x} = \frac{4}{3} \cdot \frac{\sin(4x)}{4x}.$$

Since $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$, the limit is $\frac{4}{3}$.

24. At $x = -1$, f' changes from negative to positive, so f has a local minimum. At $x = 2$, f' does not change sign, so there is no extremum. At $x = 4$, f' changes from positive to negative, so f has a local maximum.

25. Here $\Delta x = \frac{4}{n}$, so the interval has length 4, starting at 0. Since $x_i = \frac{4i}{n}$, the expression $\left(1 + \frac{4i}{n}\right)^3$ becomes $(1 + x)^3$. Therefore, the integral is

$$\int_0^4 (1 + x)^3 dx.$$

26. A particle is speeding up when velocity and acceleration have the same sign. Since

$$v(t) = (t - 1)(t - 4),$$

$v < 0$ on $(1, 4)$ and $v > 0$ on $(0, 1) \cup (4, 6)$. Also $a(t) < 0$ on $\left(0, \frac{5}{2}\right)$ and $a(t) > 0$ on $\left(\frac{5}{2}, 6\right)$.

The signs match on

$$\left(1, \frac{5}{2}\right) \text{ and } (4, 6).$$

27. Separate variables:

$$\frac{dy}{y+1} = 2x dx.$$

Integrating gives:

$$\ln|y+1| = x^2 + C,$$

so

$$y+1 = Ce^{x^2}.$$

Using $y(0) = 0$ gives $C = 1$, so

$$y = e^{x^2} - 1.$$

28. Using disks, the radius is x^2 . Therefore,

$$V = \pi \int_0^3 (x^2)^2 dx = \pi \int_0^3 x^4 dx = \pi \left[\frac{x^5}{5} \right]_0^3 = \frac{243\pi}{5}.$$

29. Use a symmetric difference quotient:

$$f'(2) \approx \frac{f(2.1) - f(1.9)}{2.1 - 1.9} = \frac{7.41 - 6.61}{0.2} = 4.0.$$

Section I, Part B

30. B	31. A	32. C	33. B	34. C
35. B	36. B	37. C	38. A	39. B
40. C	41. B	42. B		

30. Using trapezoids on each interval:

$$T = \frac{1}{2}(1)(4.2 + 5.0) + \frac{1}{2}(1.5)(5.0 + 6.3) + \frac{1}{2}(1.5)(6.3 + 5.8) + \frac{1}{2}(2)(5.8 + 4.6) + \frac{1}{2}(1)(4.6 + 3.9) = 4.6 + 8.475 + 9.075 + 10.4 + 4.25 = 36.8$$

31. First, differentiate implicitly. Then solve for $\frac{dy}{dx}$, and plug in the point (1.4, 1.7):

$$2x + x \frac{dy}{dx} + y + 6y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 6y} \approx -0.39.$$

32. Use the calculator's maximum feature on $[0, 4]$; the maximum value is approximately 2.02.

33. Calculate with symmetric difference quotient:

$$s'(5) \approx \frac{s(5.1) - s(4.9)}{5.1 - 4.9} = \frac{22.97 - 21.03}{0.2} = 9.7.$$

34. The values from the left approach 3, while the values from the right approach 4. Since the one-sided limits are different, the two-sided limit does not exist.

35. Use numerical integration:

$$A(3) = \int_0^3 \left(e^{-\frac{t^2}{2}} + \sin t \right) dt \approx 3.240.$$

36. Use a calculator to solve

$$\ln(x + 1) = \frac{x}{3}.$$

The positive intersection is 5.71144108. On $(0, 5.71144108)$, $\ln(x + 1) > \frac{x}{3}$, so the area is

$$\int_0^{5.71144108} \left(\ln(x + 1) - \frac{x}{3} \right) dx \approx 1.63.$$

37. Separate and integrate:

$$(1 + y) dy = x^2 dx.$$

$$y + \frac{y^2}{2} = \frac{x^3}{3} + C.$$

Using $y(0) = 1$, $C = \frac{3}{2}$. At $x = 2$,

$$y + \frac{y^2}{2} = \frac{8}{3} + \frac{3}{2} = \frac{25}{6}.$$

Solving using your calculator gives

$$y \approx 2.06.$$

38. Use the numerical derivate feature on your calculator at $x = 1.5$. This gives $f'(1.5) \approx -0.594$.

39. Inflection points occur where $f''(x) = 0$ and concavity changes. Solving

$$e^{-x} - 0.2x = 0$$

on $[0, 5]$ gives

$$x \approx 1.327.$$

40. Use a calculator to maximize $E(t)$ on $[0, 20]$, or solve $E'(t) = 0$ and check endpoints. The maximum occurs at $t \approx 12.142$.

41. Average value is

$$\frac{1}{2-0} \int_0^2 \sqrt{1+x^3} dx.$$

Using calculator integration,

$$\frac{1}{2} \int_0^2 \sqrt{1+x^3} dx \approx 1.621.$$

42. By the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}.$$

So

$$\frac{dV}{dt} = 48\pi(0.25) = 12\pi.$$

Section II, Part A

1. (a) Using the Power Rule and derivative of natural log:

$$R'(x) = 12 + \frac{8}{x+1} - 0.8x$$

- (b) To maximize
- $R(x)$
- on
- $[0, 20]$
- , use a calculator to solve
- $R'(x) = 0$
- .

$$12 + \frac{8}{x+1} - 0.8x = 0$$

The critical value in $(0, 20)$ is

$$x \approx 15.6023$$

Check the endpoints and the critical value. The maximum occurs at

$$x \approx 15.602$$

hundred items.

- (c) Evaluate the revenue model at the maximizing value:

$$R(15.6023) \approx 112.331.$$

Since $R(x)$ is measured in thousands of dollars, maximum revenue is approximately

$$\$112,331.$$

- (d) First, find the second derivative:

$$R''(x) = -\frac{8}{(x+1)^2} - 0.8.$$

Since

$$-\frac{8}{(x+1)^2} - 0.8 < 0$$

for all $x \in [0, 20]$, revenue is concave down on the entire interval. So revenue is increasing at a decreasing rate wherever $R'(x) > 0$. From part (b), $R'(x) = 0$ at $x \approx 15.602$. Since $R'(0) > 0$, revenue is increasing at a decreasing rate on

$$(0, 15.602).$$

2. (a) Because the time intervals are not all the same length, use trapezoids on each interval separately:

$$\begin{aligned} \int_0^9 L(t) dt &\approx \frac{1}{2}(1)(120 + 135) + \frac{1}{2}(1.5)(135 + 160) + \frac{1}{2}(1.5)(160 + 150) \\ &\quad + \frac{1}{2}(2)(150 + 110) + \frac{1}{2}(1)(110 + 95) + \frac{1}{2}(2)(95 + 80) \\ &= 127.5 + 221.25 + 232.5 + 260 + 102.5 + 175 = 1,118.75. \end{aligned}$$

$$\int_0^9 L(t) dt \approx 1,118.75 \text{ gallons}$$

- (b)
- $\int_0^9 L(t) dt \approx 1,118.75$
- gallons is the amount of water that leaks out of the reservoir in the first 9 hours.

(c) Average value:

$$L_{ave} = \frac{1}{9-0} \int_0^9 L(t) dt$$

Using the approximation from part (a):

$$L_{ave} \approx \frac{1,118.75}{9} = 124.306 \text{ gal/hr.}$$

(d) The reservoir initially contains 5,000 gallons. From part (a), about 1,118.75 gallons leak out.

$$5000 - 1118.75 = 3881.25.$$

The reservoir contains approximately 3,881.25 gallons at $t = 9$.

Section II, Part B

3. (a) Continuity requires $\lim_{x \rightarrow 1^-} f(x) = f(1) = \lim_{x \rightarrow 1^+} f(x)$.

Left-hand limit and value at $x = 1$:

$$\lim_{x \rightarrow 1^-} f(x) = f(1) = a(1)^2 + b(1) + 1 = a + b + 1.$$

Right-hand limit:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (3\sqrt{x+3} + c) = 3\sqrt{4} + c = 6 + c.$$

Set equal,

$$\begin{aligned} a + b + 1 &= 6 + c \\ c &= a + b - 5 \end{aligned}$$

- (b) Differentiability requires matching derivatives from the left and right at $x = 1$.

Left derivative ($x \leq 1$):

$$\begin{aligned} f'(x) &= 2ax + b \\ f'_-(1) &= 2a + b. \end{aligned}$$

Right derivative ($x > 1$):

$$\begin{aligned} f'(x) &= 3 \cdot \frac{1}{2\sqrt{x+3}} = \frac{3}{2\sqrt{x+3}} \\ f'_+(1) &= \frac{3}{2\sqrt{4}} = \frac{3}{4}. \end{aligned}$$

Set equal:

$$2a + b = \frac{3}{4}.$$

- (c) If f is differentiable at $x = 1$, then slope is $f'(1) = \frac{3}{4}$ (from part (b)), and the point is $(1, f(1))$ where $f(1) = a + b + 1$ (from part (a)).

Using point-slope form:

$$\begin{aligned} y - (a + b + 1) &= \frac{3}{4}(x - 1) \\ y &= \frac{3}{4}x + a + b + \frac{1}{4}. \end{aligned}$$

- (d) Using part (b), we know $2a + b = \frac{3}{4}$. Therefore, if $a = 2$:

$$4 + b = \frac{3}{4}$$

$$b = -\frac{13}{4}.$$

Now use the continuity condition from part (a):

$$c = a + b - 5.$$

Substitute $a = 2$ and $b = -\frac{13}{4}$:

$$c = 2 - \frac{13}{4} - 5 = -\frac{25}{4}.$$

$$b = -\frac{13}{4}, \quad c = -\frac{25}{4}$$

4. (a) First, find the derivative:

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1).$$

Critical values occur where $f'(x) = 0$: $x = -1$ and $x = 3$.

$$f'(x) = 3(x - 3)(x + 1), \text{ critical values: } x = -1, 3$$

- (b) Use the sign of $f'(x) = 3(x - 3)(x + 1)$. This can be shown with a gradient diagram.

x	-2	-1	0	3	4
Sign of $f'(x)$	+	0	-	0	+
Direction of $f(x)$					

f is increasing on $(-\infty, -1) \cup (3, \infty)$, and decreasing on $(-1, 3)$.

- (c) First, find the second derivative:

$$f''(x) = 6x - 6 = 6(x - 1).$$

Inflection points may occur where $f''(x) = 0$: $x = 1$. Since $f''(x)$ changes sign from negative (when $x < 1$) to positive (when $x > 1$), f has an inflection point at $x = 1$.

- (d) Since f is a polynomial, it is continuous on $[0, 4]$ and differentiable on $(0, 4)$, so the Mean Value Theorem applies.

Compute the average rate of change:

$$\frac{f(4) - f(0)}{4 - 0} = -\frac{-16}{4} = \frac{-16 - 4}{4} = -5.$$

Now set

$$f'(c) = -5.$$

$$3c^2 - 6c - 9 = -5$$

$$3c^2 - 6c - 4 = 0.$$

Use the quadratic formula:

$$c = \frac{6 \pm \sqrt{36 + 48}}{6} = \frac{6 \pm \sqrt{84}}{6} = \frac{6 \pm 2\sqrt{21}}{6} = 1 \pm \frac{\sqrt{21}}{3}.$$

The only value in $(0, 4)$ is

$$c = 1 + \frac{\sqrt{21}}{3}.$$

5. (a) Since

$$g(2) = \int_0^2 f(t) dt,$$

we need the signed area from $t = 0$ to $t = 2$.

The line segment goes from $(0, 3)$ to $(2, -1)$. Its equation is

$$f(t) = 3 - 2t.$$

So

$$\begin{aligned} g(2) &= \int_0^2 (3 - 2t) dt. \\ g(2) &= [3t - t^2] \Big|_0^2 = 6 - 4 = 2. \end{aligned}$$

(b) By the Fundamental Theorem of Calculus,

$$g'(x) = f(x).$$

On $[5, 7]$, f is the line from $(5, -1)$ to $(7, 3)$ with slope $\frac{3 - (-1)}{7 - 5} = 2$.

So, its equation is

$$y - (-1) = 2(x - 5)$$

$$f(x) = 2x - 11.$$

Therefore,

$$g'(6) = f(6) = 2(6) - 11 = 1.$$

$$g'(6) = 1$$

(c) g is decreasing where $g'(x) < 0$, i.e., where $f(x) < 0$. From part (a), the first line segment is $f(x) = 3 - 2x$.

Set $f(x) = 0$:

$$3 - 2x = 0$$

$$x = \frac{3}{2}.$$

The graph remains below the x -axis from $x = \frac{3}{2}$ until the last segment crosses the x -axis.

On the last segment from $(5, -1)$ to $(7, 3)$, $f(x) = 2x - 11$.

Set $f(x) = 0$:

$$2x - 11 = 0$$

$$x = \frac{11}{2}.$$

Combining these:

$$g \text{ is decreasing on } \left(\frac{3}{2}, \frac{11}{2}\right)$$

(d) The absolute minimum of g on a closed interval occurs at endpoints or where $g'(x) = 0$.

Since $g'(x) = f(x)$, candidates are $x = 0$, $x = \frac{3}{2}$, $x = \frac{11}{2}$, and $x = 7$.

Compute $g(x)$ at these points:

- $g(0) = 0$
- $g\left(\frac{3}{2}\right) = \int_0^{3/2} f(t) \, dt$

At $x = \frac{3}{2}$, g has accumulated positive area from 0 to $\frac{3}{2}$, so it can't be a minimum.

- $g\left(\frac{11}{2}\right)$, use the accumulated area.

From $x = 0$ to 2:

$$g(3) = 2.$$

From $x = 2$ to 5, $f(x) = -1$, so

$$\int_2^5 f(x) \, dx = -3.$$

Thus

$$g(5) = 2 - 3 = -1.$$

From $x = 5$ to $\frac{11}{2}$, the graph is below the x -axis and forms a triangle with base $\frac{1}{2}$ and height 1:

$$-\frac{1}{2}\left(\frac{1}{2}\right)(1) = -\frac{1}{4}.$$

Thus,

$$g\left(\frac{11}{2}\right) = -1 - \frac{1}{4} = -\frac{5}{4}.$$

- $g(7) = \int_0^7 f(t) dt$

Since g decreases until $x = \frac{11}{2}$ and then increases after that, $g(7) > g\left(\frac{11}{2}\right)$ and can't be a minimum.

Compare $g(0)$, $g\left(\frac{3}{2}\right)$, $g\left(\frac{11}{2}\right)$, and $g(7)$.

The smallest value is $g\left(\frac{11}{2}\right) = -\frac{5}{4}$.

The absolute minimum value of g on $[0, 7]$ is $-\frac{5}{4}$ (at $x = \frac{11}{2}$).

6. (a) Given

$$y = 2 + 3e^{x^2/2},$$

differentiate:

$$\frac{dy}{dx} = 3e^{x^2/2} \cdot x$$

$$\frac{dy}{dx} = 3xe^{x^2/2}$$

Now compute $x(y - 2)$:

$$x(y - 2) = x(2 + 3e^{x^2/2} - 2)$$

$$x(y - 2) = 3xe^{x^2/2}.$$

Therefore,

$$\frac{dy}{dx} = x(y - 2),$$

so the function satisfies the differential equation.

Also,

$$y(0) = 2 + 3e^0 = 2 + 3 = 5.$$

So it satisfies the initial condition.

(b) Separate the variables:

$$\frac{dy}{y-2} = x \, dx$$

Integrate:

$$\int \frac{dy}{y-2} = \int x \, dx$$

$$\ln |y-2| = \frac{1}{2}x^2 + C_1$$

Isolate y :

$$y-2 = e^{\frac{1}{2}x^2+C_1} = Ce^{\frac{1}{2}x^2}$$

$$y(x) = Ce^{\frac{1}{2}x^2} + 2$$

(c) Plug in $x = 0, y = 5$:

$$5 = Ce^{\frac{1}{2}(0)^2} + 2$$

$$5 = Ce^0 + 2$$

$$3 = Ce^0$$

$$C = 3$$

So,

$$f(x) = 3e^{\frac{1}{2}x^2} + 2.$$

And

$$f(2) = 3e^2 + 2$$

(d) The solution is increasing where $\frac{dy}{dx} > 0$.

Using the differential equation:

$$\frac{dy}{dx} = x(y-2).$$

For the particular solution, $y - 2 = 3e^{\frac{1}{2}x^2}$.

Since

$$3e^{\frac{1}{2}x^2} > 0$$

for all real x , the sign of $\frac{dy}{dx}$ depends only on x .

Therefore, $\frac{dy}{dx} > 0$ when $x > 0$.

The particular solution is increasing on $(0, \infty)$.